Riemann Solvers in General Relativistic Hydrodynamics

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Abstract

Our contribution concerns with the numerical solution of the 3D general relativistic hydrodynamical system of equations within the framework of the $\{3 + 1\}$ formalism. We summarize the theoretical ingredients which are necessary in order to build up a numerical scheme based on the solution of local Riemann problems. Hence, the full spectral decomposition of the Jacobian matrices of the system, i.e., the eigenvalues and the right and left eigenvectors, is explicitly shown. An alternative approach consists in using any of the special relativistic Riemann solvers recently developed for describing the evolution of special relativistic flows. Our proposal relies on a local change of coordinates in terms of which the spacetime metric is locally Minkowskian and permits an accurate description of numerical general relativistic hydrodynamics.

1 Introduction

Astrophysical scenarios involving relativistic flows have drawn the attention and efforts of many researchers since the pioneering studies of May and White [16] and Wilson [23]. Relativistic jets, accretion onto compact objects (in X-ray binaries or in the inner regions of active galactic nuclei), stellar core collapse, coalescing compact binaries (neutron star and/or black holes) and recent models of formation of Gamma-ray bursts (GRBs) are examples of systems in which the evolution of matter is described within the frame of the theory of relativity (special or general).

Since 1991 [14] the use of Riemann solvers in relativistic hydrodynamics has proved successful in handling complex flows, with high Lorentz factors and strong shocks, superseding traditional methods which failed to describe ultrarelativistic flows [17]. Exploiting the hyperbolic and conservative character of the relativistic hydrodynamical equations, we proposed how to extend *modern high-resolution shock-capturing* (HRSC) methods to the relativistic case, first in one-dimensional calculations [14], and, later on, we extended them to multidimensional special relativistic [7] and multidimensional general relativistic hydrodynamics [4]. We made use of a linearized Riemann solver based on the *spectral decomposition* of the Jacobian matrices of the system.

Unlike the case of classical fluid dynamics the use of HRSC techniques in the frame of relativistic fluid dynamics is very recent and has yet to cover the full set of possible applications. Up to now, the most interesting astrophysical applications have involved the simulation of extragalactic relativistic jets (see [1] and [11] for, respectively, relativistic 3D-hydro and 2D-magnetohydro calculations). Recently, some studies on the morphology of accreting flows onto moving black holes have being carried out (see [8] and references therein) using a multidimensional general relativistic hydrocode. A very promising application of HRSC techniques in the frame of general relativistic magnetohydrodynamics has been used recently to simulate the formation of relativistic jets from black holes magnetized accretion disks [12].

At present, to develop robust and accurate general relativistic hydrocodes is a challenge in the field of Relativistic Astrophysics. A general relativistic hydrocode is a useful research tool for studying flows which evolve in a background spacetime. Furthermore, when appropriately coupled with Einstein equations, such a general relativistic hydrocode is crucial to model the evolution of matter in a dynamical spacetime. The coupling between geometry and matter arises through the sources of the corresponding system of equations. Such a marriage between numerical relativity and numerical relativistic hydrodynamics could be useful, for example, to analyze the dynamics (and the physics) of coalescing compact binaries. These are one of the most promising sources of gravitational radiation to be detected by the near future Earth-based laser-interferometer observatories of gravitational waves.

2 The equations of general relativistic hydrodynamics as a hyperbolic system of conservation laws

The evolution of a relativistic fluid is governed by a system of equations which summarize *local conservation laws*: the local conservation of baryon number, $\nabla \cdot \mathbf{J} = 0$, and the local conservation of energy-momentum, $\nabla \cdot \mathbf{T} = 0$ ($\nabla\cdot$ stands for the covariant divergence).

If $\{\partial_\mathbf{t}, \partial_\mathbf{i}\}$ define the coordinate basis of 4-vectors which are tangents to the corresponding coordinate curves, then, the *current of rest-mass*, $\mathbf{J}$, and the *energy-momentum tensor*, $\mathbf{T}$, for a perfect fluid, have the components: $J^\mu = \rho u^\mu$, and $T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$, respectively, ρ being the rest-mass density, p the pressure and h the specific enthalpy, defined by $h = 1 + \varepsilon + p/\rho$, where ε is the specific internal energy. u^μ is the four-velocity of the fluid and $g_{\mu\nu}$ defines the metric of the spacetime $\mathcal{M}$ where the fluid evolves. As usually, Greek (Latin) indices run from 0 to 3 (1 to 3) – or, alternatively, they stand for the general coordinates $\{t, x, y, z\}$ ($\{x, y, z\}$) – and the system of units is the so-called geometrized ($c = G = 1$).

An equation of state $p = p(\rho, \varepsilon)$ closes, as usual, the system. Accordingly, the local sound velocity c_s satisfies: $hc_s^2 = \chi + (p/\rho^2)\kappa$, with $\chi = \partial p/\partial\rho|_\epsilon$ and $\kappa = \partial p/\partial\epsilon|_\rho$.

Following [4], let $\mathcal{M}$ be a general spacetime, described by the four dimensional metric tensor $g_{\mu\nu}$. According to the $\{3+1\}$ formalism, the metric is split into the objects α (*lapse*), β^i (*shift*) and γ_{ij}, keeping the line element in the form:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j \tag{1}$$

If $\mathbf{n}$ is a unit timelike vector field normal to the spacelike hypersurfaces Σ_t (t = const.), then, by definition of α and β^i is: $\partial_t = \alpha \mathbf{n} + \beta^i \partial_i$, with $\mathbf{n} \cdot \partial_i = 0$, $\forall i$. Observers, $\mathcal{O}_E$, at rest in the slice Σ_t, i.e., those having $\mathbf{n}$ as four-velocity (*Eulerian observers*), measure the following velocity of the fluid

$$v^i = \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha} \tag{2}$$

where $W \equiv -(\mathbf{u} \cdot \mathbf{n}) = \alpha u^t$, the Lorentz factor, satisfies $W = (1 - \mathrm{v}^2)^{-1/2}$ with $\mathrm{v}^2 = v_i v^i$ ($v_i = \gamma_{ij} v^j$).

Let us define a basis adapted to the observer $\mathcal{O}_E$, $\mathbf{e}_{(\mu)} = \{\mathbf{n}, \partial_i\}$, and the following five four-vector fields $\{\mathbf{J},\ \mathbf{T}\cdot\mathbf{n},\ \mathbf{T}\cdot\partial_1,\ \mathbf{T}\cdot\partial_2,\ \mathbf{T}\cdot\partial_3\}$. Hence, the above system of equations of general relativistic hydrodynamics (GRH) can be written

$$\nabla \cdot \mathbf{A} = s, \tag{3}$$

where $\mathbf{A}$ denotes any of the above 5 vector fields, and s is the corresponding source term.

The set of *conserved variables* gathers those quantities which are directly measured by $\mathcal{O}_E$, i.e., the rest-mass density (D), the momentum density in the j-direction (S_j) and the total energy density (E). In terms of the *primitive variables* $\mathbf{w} = (\rho, v_i, \varepsilon)$ ($v_i = \gamma_{ij} v^j$) they are

$$D = \rho W \quad , \quad S_j = \rho h W^2 v_j \quad , \quad E = \rho h W^2 - p \tag{4}$$

Taking all the above relations together, the fundamental system to be considered for numerical applications is

$$\frac{1}{\sqrt{-g}} \left(\frac{\partial \sqrt{\gamma} \mathbf{F}^0(\mathbf{w})}{\partial x^0} + \frac{\partial \sqrt{-g} \mathbf{F}^i(\mathbf{w})}{\partial x^i} \right) = \mathbf{s}(\mathbf{w}) \tag{5}$$

where the quantities $\mathbf{F}^\alpha(\mathbf{w})$ are

$$\mathbf{F}^0(\mathbf{w}) = (D, S_j, \tau) \tag{6}$$

$$\mathbf{F}^i(\mathbf{w}) = \left(D\left(v^i - \frac{\beta^i}{\alpha}\right), S_j \left(v^i - \frac{\beta^i}{\alpha}\right) + p\delta^i_j, \tau\left(v^i - \frac{\beta^i}{\alpha}\right) + pv^i \right) \tag{7}$$

and the corresponding sources $\mathbf{s}(\mathbf{w})$ are

$$\mathbf{s}(\mathbf{w}) = \left(0, T^{\mu\nu} \left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu} g_{\delta j} \right), \alpha \left(T^{\mu 0} \frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu} \Gamma^0_{\nu\mu} \right) \right) \tag{8}$$

τ being $\tau \equiv E - D$, and $g \equiv \det(g_{\mu\nu})$ is such that $\sqrt{-g} = \alpha\sqrt{\gamma}$ ($\gamma \equiv \det(\gamma_{ij})$)

2.1 Linearized Riemann Solvers and Characteristic fields

Modern HRSC schemes use the characteristic structure of the hyperbolic system of conservation laws. In many Godunov-type schemes, the characteristic structure is used to compute either an exact or an approximate solution to a sequence of Riemann problems at each cell interface. In characteristic based methods the characteristic structure is used to compute the local characteristic fields, which define the directions along which the characteristic variables propagate. In both these approaches, the characteristic decomposition of the Jacobian matrices of the nonlinear system of conservation laws is important, not only because it is one of the key ingredients in the design of the numerical flux at the interfaces, but because experience has shown that it facilitates a robust upgrading of the order of a numerical scheme.

The three 5×5-Jacobian matrices $\mathcal{B}^i$ associated to system (5) are

$$\mathcal{B}^i = \alpha \frac{\partial \mathbf{F}^i}{\partial \mathbf{F}^0}. \tag{9}$$

The *eigenvalues* of $\mathcal{B}^x$ are

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)} \tag{10}$$

which defines the *material waves*, and two other, $\lambda_\pm$, associated with the *acoustic waves*

$$\lambda_\pm = \frac{\alpha}{1 - v^2 c_s^2} \left\{ v^x (1 - c_s^2) \pm \right.$$
$$\left. \pm c_s \sqrt{(1 - v^2)[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)]} \right\} - \beta^x \tag{11}$$

A complete set of *right-eigenvectors* is

$$\mathbf{r}_\pm = \begin{bmatrix} 1 \\ hW \left(v_x - \dfrac{v^x - \Lambda^x_\pm}{\gamma^{xx} - v^x \Lambda^x_\pm} \right) \\ hW v_y \\ hW v_z \\ \dfrac{hW(\gamma^{xx} - v^x v^x)}{\gamma^{xx} - v^x \Lambda^x_\pm} - 1 \end{bmatrix}, \quad \mathbf{r}_{\mathbf{0},\mathbf{1}} = \begin{bmatrix} \dfrac{\mathcal{K}}{hW} \\ v_x \\ v_y \\ v_z \\ 1 - \dfrac{\mathcal{K}}{hW} \end{bmatrix}$$

$$\mathbf{r_{0,2}} = \begin{bmatrix} Wv_y \\ h\left(\gamma_{xy} + 2W^2 v_x v_y\right) \\ h\left(\gamma_{yy} + 2W^2 v_y v_y\right) \\ h\left(\gamma_{zy} + 2W^2 v_z v_y\right) \\ Wv_y(2hW-1) \end{bmatrix} \quad , \quad \mathbf{r_{0,3}} = \begin{bmatrix} Wv_z \\ h\left(\gamma_{xz} + 2W^2 v_x v_z\right) \\ h\left(\gamma_{yz} + 2W^2 v_y v_z\right) \\ h\left(\gamma_{zz} + 2W^2 v_z v_z\right) \\ Wv_z(2hW-1) \end{bmatrix}$$

The corresponding set of *left-eigenvectors* is

$$\mathbf{l}_{0,1} = \frac{W}{\mathcal{K}-1}\left(h - W, Wv^x, Wv^y, Wv^z, -W\right)$$

$$\mathbf{l}_{0,2} = \frac{1}{h\xi}\begin{bmatrix} -\gamma_{zz}v_y + \gamma_{yz}v_z \\ v^x(\gamma_{zz}v_y - \gamma_{yz}v_z) \\ \gamma_{zz}(1 - v_x v^x) + \gamma_{xz}v_z v^x \\ -\gamma_{yz}(1 - v_x v^x) - \gamma_{xz}v_y v^x \\ -\gamma_{zz}v_y + \gamma_{yz}v_z \end{bmatrix}, \mathbf{l}_{0,3} = \frac{1}{h\xi}\begin{bmatrix} -\gamma_{yy}v_z + \gamma_{zy}v_y \\ v^x(\gamma_{yy}v_z - \gamma_{zy}v_y) \\ -\gamma_{zy}(1 - v_x v^x) - \gamma_{xy}v_z v^x \\ \gamma_{yy}(1 - v_x v^x) + \gamma_{xy}v_y v^x \\ -\gamma_{yy}v_z + \gamma_{zy}v_y \end{bmatrix}$$

$$\mathbf{l}_{\mp} = (\pm 1)\frac{h^2}{\Delta}\begin{bmatrix} hW\mathcal{V}^x_{\pm}\xi + l^{(5)}_{\mp} \\ \Gamma_{xx}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K}-1)\mathcal{V}^x_{\pm}(W^2 v^x\xi - \Gamma_{xx}v^x) \\ \Gamma_{xy}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K}-1)\mathcal{V}^x_{\pm}(W^2 v^y\xi - \Gamma_{xy}v^x) \\ \Gamma_{xz}(1 - \mathcal{K}\tilde{\mathcal{A}}^x_{\pm}) + (2\mathcal{K}-1)\mathcal{V}^x_{\pm}(W^2 v^z\xi - \Gamma_{xz}v^x) \\ (1-\mathcal{K})[-\gamma v^x + \mathcal{V}^x_{\pm}(W^2\xi - \Gamma_{xx})] - \mathcal{K}W^2\mathcal{V}^x_{\pm}\xi \end{bmatrix}$$

where the following auxiliary quantities and relations have been introduced:

$$\Lambda^i_{\pm} \equiv \tilde{\lambda}_{\pm} + \tilde{\beta}^i \quad , \quad \tilde{\lambda} \equiv \lambda/\alpha \quad , \quad \tilde{\beta}^i \equiv \beta^i/\alpha \tag{12}$$

$$\mathcal{K} \equiv \frac{\tilde{\kappa}}{\tilde{\kappa} - c_s^2} \quad , \quad \tilde{\kappa} \equiv \kappa/\rho \tag{13}$$

$$\mathcal{C}^x_\pm \equiv v_x - \mathcal{V}^x_\pm \;\; , \;\; \mathcal{V}^x_\pm \equiv \frac{v^x - \Lambda^x_\pm}{\gamma^{xx} - v^x\Lambda^x_\pm} \tag{14}$$

$$\tilde{\mathcal{A}}^x_\pm \equiv \frac{\gamma^{xx} - v^x v^x}{\gamma^{xx} - v^x\Lambda^x_\pm} \tag{15}$$

$$\Delta \equiv h^3 W(\mathcal{K}-1)(\mathcal{C}^x_+ - \mathcal{C}^x_-)\xi \tag{16}$$

$$\xi \equiv \Gamma_{xx} - \gamma v^x v^x \;\; , \;\; \Gamma_{xx} = \gamma_{yy}\gamma_{zz} - \gamma^2_{yz} \;\; , \;\; \ldots \tag{17}$$

Symmetry arguments allow one to obtain the spectral decomposition in the other spatial directions. The corresponding expressions in Special Relativity [5] are easily covered. The above full spectral decomposition provides the user with the technical ingredients needed to develop state-of-the-art, upwind-based HRSC codes for numerical relativistic hydrodynamics. In 3D general-relativistic applications, and depending on the particular Riemann solver or flux formula used, the knowledge of the analytical values of the left-eigenvectors could be crucial in the efficiency of the code (see [2]).

3 Special Relativistic Riemann Solvers in General Relativistic Hydrodynamics

Up to now, only a small number of papers have considered the extension of HRSC methods to GRH using linearized Riemann Solvers or flux formulae ([14] and [21] for 1D problems, [4], [9] and [18]). or deriving explicitly an extension of Roe's Riemann solver to GRH ([6]).

In [20] we have answered the following basic question (suggested in [3] and [13]): Is it possible to obtain a general procedure that allows one to take advantage of Special Relativistic Riemann Solvers (SRRS) to generate numerical solutions describing the evolution of relativistic flows in strong gravitational fields?

The affirmative answer relies on a *local change of coordinates*, at each numerical interface, in terms of which the spacetime metric is locally Minkowskian. Our procedure, hence, follows analogous trends to those used in classical fluid dynamics to solve Riemann problems in general curvilinear coordinates.

Let us consider the integral form of the system of equations (3) on a four–dimensional volume Ω, with three–dimensional boundary $\partial\Omega$, and apply Gauss theorem to obtain the corresponding balance equation

$$\int_{\partial\Omega} \mathbf{A} \cdot d^3\mathbf{\Sigma} = \int_\Omega s d\Omega. \tag{18}$$

For numerical applications, we choose volume Ω as the one bounded by the coordinate hypersurfaces $\{\Sigma_{x^\alpha}, \Sigma_{x^\alpha+\Delta x^\alpha}\}$. Hence, the time variation of the mean value of A^0,

$$\overline{A}^0 = \frac{1}{\Delta\mathcal{V}} \int_{x^1}^{x^1+\Delta x^1} \int_{x^2}^{x^2+\Delta x^2} \int_{x^3}^{x^3+\Delta x^3} \sqrt{-g}\, A^0 dx^1 dx^2 dx^3, \tag{19}$$

within the spatial volume

$$\Delta\mathcal{V} = \int_{x^1}^{x^1+\Delta x^1} \int_{x^2}^{x^2+\Delta x^2} \int_{x^3}^{x^3+\Delta x^3} \sqrt{-g} dx^1 dx^2 dx^3, \tag{20}$$

can be obtained from

$$\begin{aligned}(\overline{A}^0 \Delta\mathcal{V})_{t+\Delta t} = \quad & (\overline{A}^0 \Delta\mathcal{V})_t + \int_\Omega s d\Omega \; - \\ & \left(\int_{\Sigma_{x^1}} \mathbf{A} \cdot d^3\mathbf{\Sigma} + \int_{\Sigma_{x^1+\Delta x^1}} \mathbf{A} \cdot d^3\mathbf{\Sigma} + \right. \\ & \int_{\Sigma_{x^2}} \mathbf{A} \cdot d^3\mathbf{\Sigma} + \int_{\Sigma_{x^2+\Delta x^2}} \mathbf{A} \cdot d^3\mathbf{\Sigma} + \\ & \left. \int_{\Sigma_{x^3}} \mathbf{A} \cdot d^3\mathbf{\Sigma} + \int_{\Sigma_{x^3+\Delta x^3}} \mathbf{A} \cdot d^3\mathbf{\Sigma} \right). \end{aligned} \tag{21}$$

In order to advance in time, the volume and surface integrals on the right hand side have to be evaluated. We have applied HRSC to calculate the $\mathbf{A}$ vector fields by solving local Riemann problems combined with monotonized cell reconstruction techniques.

According to the Equivalence Principle, physical laws in a *local inertial frame* of a curved spacetime have the same form as in special relativity (see, e.g., Schutz [22]). Locally flat (or geodesic) systems of coordinates, in which the metric is brought into the Minkowskian form up to second order terms, are the realization of these local inertial frames. However, whereas the coordinate transformation leading to locally flat coordinates involves second order terms, locally Minkowskian coordinates are obtained by a linear transformation. This fact is of crucial importance when exploiting the self-similar character of the solution of the Riemann problems set up across the coordinate surfaces.

Hence, we propose to perform a coordinate transformation to locally Minkowskian coordinates at each numerical interface assuming that the solution of the Riemann problem is one in special relativity and planar symmetry. This last assumption is equivalent to the approach followed in classical fluid dynamics, when using the solution of Riemann problems in slab symmetry for problems in cylindrical or spherical coordinates, which breaks down near the singular points (*e.g.* the polar axis in cylindrical coordinates). Analogously to classical fluid dynamics, the numerical error will depend on the magnitude of the Christoffel symbols, which might be large whenever huge gradients or large temporal variations of the gravitational field are present. Finer grids and improved time advancing methods will be required in those regions.

In the rest of this section we will focus on the evaluation of the first flux integral in Eq. (21).

$$\int_{\Sigma_{x^1}} \mathbf{A} \cdot d^3\mathbf{\Sigma} = \int_{\Sigma_{x^1}} A^1 \sqrt{-g}\, dx^0 dx^2 dx^3 \tag{22}$$

To begin, we will express the integral on a basis $\mathbf{e}_{\hat{\alpha}}$ with $\mathbf{e}_{\hat{0}} \equiv \mathbf{n}$ and $\mathbf{e}_{\hat{i}}$ forming an orthonormal basis in the plane orthogonal to $\mathbf{n}$ with $\mathbf{e}_{\hat{1}}$ normal to the surface Σ_{x^1} and $\mathbf{e}_{\hat{2}}$ and $\mathbf{e}_{\hat{3}}$ tangent to that surface. The vectors of this basis verify $\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ with $\eta_{\hat{\alpha}\hat{\beta}}$ the Minkowski metric (in the following, caret subscripts will refer to vector components in this basis).

Denoting by x_0^α the coordinates at the center of the interface at time t, we introduce the following locally Minkowskian coordinate system

$$x^{\hat{\alpha}} = M^{\hat{\alpha}}_{\ \alpha}\,(x^\alpha - x_0^\alpha), \tag{23}$$

where the matrix $M^{\hat{\alpha}}_\alpha$ is given by $\partial_\alpha = M^{\hat{\alpha}}_\alpha \mathbf{e}_{\hat{\alpha}}$, calculated at x_0^α. In this system of coordinates the equations of general relativistic hydrodynamics transform into the equations of special relativistic hydrodynamics, in Cartesian coordinates, but with non-zero sources, and the flux integral (22) reads

$$\int_{\Sigma_{x^1}} (A^{\hat{1}} - \frac{\beta^{\hat{1}}}{\alpha} A^{\hat{0}}) \sqrt{-\hat{g}}\, dx^{\hat{0}} dx^{\hat{2}} dx^{\hat{3}} \tag{24}$$

with $\sqrt{-\hat{g}} = 1 + \mathcal{O}(x^{\hat{\alpha}})$, where we have taken into account that, in the coordinates $x^{\hat{\alpha}}$, Σ_{x^1} is described by the equation $x^{\hat{1}} - \frac{\beta^{\hat{1}}}{\alpha} x^{\hat{0}} = 0$ (with $\beta^{\hat{i}} = M^{\hat{i}}_i \beta^i$), where the metric elements β^1 and α are calculated at x_0^α. Therefore, this surface is not at rest but moves with *speed* $\beta^{\hat{1}}/\alpha$.

At this point, all the theoretical work on SRRS developed in recent years, can be exploited. The quantity in parenthesis in (24) represents the numerical flux across Σ_{x^1}, which can, now, be calculated by solving the special relativistic Riemann problem defined with the values at the two sides of Σ_{x^1} of two independent thermodynamical variables (namely, the rest mass density ρ and the specific internal energy ϵ) and the components of the velocity in the orthonormal spatial basis $v^{\hat{i}}$ ($v^{\hat{i}} = M^{\hat{i}}_i v^i$). Although most linearized Riemann solvers provide the numerical fluxes for surfaces at rest, it is easy to apply them to moving surfaces, relying on the conservative and hyperbolic character of the system of equations (as in [10]).

Once the Riemann problem has been solved, by means of any linearized or exact SRRS, we can take advantage of the self-similar character of the solution of the Riemann problem, which makes it constant on the surface Σ_{x^1} simplifying the calculation of the above integral enormously (24):

$$\int_{\Sigma_{x^1}} \mathbf{A} \cdot d^3\mathbf{\Sigma} = (A^{\hat{1}} - \frac{\beta^{\hat{1}}}{\alpha} A^{\hat{0}})^* \int_{\Sigma_{x^1}} \sqrt{-\hat{g}}\, dx^{\hat{0}} dx^{\hat{2}} dx^{\hat{3}} \tag{25}$$

where the superscript (*) stands for the value on Σ_{x^1} obtained from the solution of the Riemann problem. The integral in the right hand side of (24) is the area of the surface Σ_{x^1} and can be expressed in terms of the original coordinates as

$$\int_{\Sigma_{x^1}} \sqrt{\gamma^{11}} \sqrt{-g}\, dx^0 dx^2 dx^3 \tag{26}$$

which can be evaluated for a given metric.

Finally, notice that the numerical fluxes defined in (24) correspond to the vector fields $\mathbf{A} = \{\mathbf{J},\, \mathbf{T}\cdot\mathbf{n},\, \mathbf{T}\cdot e_{\hat{1}},\, \mathbf{T}\cdot e_{\hat{2}},\, \mathbf{T}\; e_{\hat{3}}\}$. Thus the additional relation

$$\mathbf{T}\cdot\partial_i = M_i^{\hat{j}}(\mathbf{T}\cdot\mathbf{e}_{\hat{j}}) \tag{27}$$

has to be used for the momentum equations.

The interested reader can address reference [20] for details on the testing and calibration of our procedure. The additional computational cost of the approach is completely negligible. The procedure has a large potentiality and can be applied to other systems of conservation laws, such as magneto-hydrodynamics (MHD), making possible to solve the general relativistic MHD equations using the corresponding Riemann solvers developed for the special relativistic case.

4 Conclusions

An appropriate conservative formulation for the equations, together with the knowledge of the characteristic fields associated to the system, define the starting point for using HRSC schemes. The spectral decomposition of the Jacobian matrices, corresponding to the fluxes in each spatial direction, is used in the numerical flux computation and, moreover, it is potentially interesting in allowing an extensive range of application of HRSC methods with different approximate Riemann solvers or flux formulae.

The procedure outlined in Section §3 is –from the computational point of view – very cheap, since it involves a linear change of coordinates. It has a large potentiality and can be applied to other systems of conservation laws, as magnetohydrodynamics, giving a very useful numerical tool to solve the general relativistic MHD equations using the corresponding Riemann solvers developed for the special relativistic case. In particular, it is possible to use the exact solution of the special relativistic Riemann problem [15], [19].

The astrophysical applications foreseen in the present and near future include the study of jet formation, multidimensional stellar core collapse, gamma-ray bursts and the coalescence of compact binaries. HRSC methods can, without doubt, be used successfully to tackle these scenarios, and acquire the prestige they have already earned in the simulation of relativistic jets and accretion flows around compact objects.

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